Finitary and Infinitary Mathematics,
the Possibility of Possibilities and the Definition of Probabilities. *

Matthew J. Donald

The Cavendish Laboratory, JJ Thomson Avenue,
Cambridge CB3 0HE, Great Britain.

e-mail: mjd1014@cam.ac.uk

web site: http://www.bss.phy.cam.ac.uk/~mjd1014

Abstract Some relations between physics and finitary and infinitary mathematics are explored in the context of a many-minds interpretation of quantum theory. The analogy between mathematical "existence" and physical "existence" is considered from the point of view of philosophical idealism. Some of the ways in which infinitary mathematics arises in modern mathematical physics are discussed. Empirical science has led to the mathematics of quantum theory. This in turn can be taken to suggest a picture of reality involving possible minds and the physical laws which determine their probabilities. In this picture, finitary and infinitary mathematics play separate roles. It is argued that mind, language, and finitary mathematics have similar prerequisites, in that each depends on the possibility of possibilities. The infinite, on the other hand, can be described but never experienced, and yet it seems that sets of possibilities and the physical laws which define their probabilities can be described most simply in terms of infinitary mathematics.

This is an extended version of a talk given to an audience of mixed backgrounds in a philosophy of mathematics seminar series in Cambridge. My aim is to explore some ideas about the relation between mathematics and reality in the context of the version of the many-minds interpretation of quantum theory which I have been developing for many years. I have been led to my present radical views on reality as a result of taking the mathematics of quantum theory all too seriously. The justification of these views requires that the structure of reality is governed by mathematical law. But what sort of mathematics is ultimately required?

Philosophers of mathematics have made many suggestions about the nature of mathematics and almost all of them have some validity. Mathematics is useful if-thenism and a language game and a social practice and a means of discovering eternal truths and reducible to set theory; speaking of the existence of the set of real numbers as a completed infinite totality is both useful and dangerous; numbers are structures which obey the rules which would be obeyed by ratios of physical lengths in a physical Euclidean geometry or the rules which would be obeyed by any abstract structure given appropriate axioms. Such a wide range of good ideas seems to arise because there are so many different levels at which mathematics is important. Unfortunately, philosophers of mathematics are often tempted to be dogmatic and to claim that mathematics is nothing but useful if-thenism, or nothing but a language game, or nothing but a social practice, or nothing but set theory; or only certain types of

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construction are to be allowed; or that numbers are nothing but sets, or nothing but ratios of lengths. Such dogmatic claims are rarely placed in the context of the sort of thorough analysis of the nature of reality which should surely be required to justify them, especially given the importance of mathematics to physics. The dogmatic claim I wish to investigate in this paper is the idea that, ultimately, only finitary mathematics is indispensable.

The idea in the philosophy of mathematics, that a certain aspect of mathematics is "indispensable", is, broadly, the idea that that aspect is required for science and that therefore we ought to accept its reality. "Finitary mathematics" is mathematics which can be expressed without invoking infinite sets. This will be left as a fairly vague notion here. For example, the fact that $\sqrt{2}$ is irrational is infinitary, but this does not imply that all circumstances in which we refer to a "number" r such that $r^2 = 2$ involve infinitary mathematics. The question of how this vagueness might be eliminated will be ignored, because we shall focus on the idea of "ultimately indispensable" and explore instead some questions about the possibility of a serious and plausible complete scientific understanding based on a finite ontology.

My analysis of quantum theory is one motivation for exploring these questions. Quantum theory has led me to a form of philosophical idealism. According to this, mind is the primary aspect of reality and each individual mind is finite. Nevertheless, infinitary mathematics still seems to be needed for the definition of probabilities of individual mental states. My personal goals in the work leading up to this paper have been to try to understand the relationship between the finitary and the infinitary in this context and to consider whether the use of infinitary mathematics can be avoided. However, although it is often illuminating, and may ultimately be necessary, to think about philosophical questions in a specific framework, the details of my interpretation of quantum theory are certainly not relevant to the present paper and many of the questions raised seem to me to be significant in a range of different contexts.

For example, mathematical physics certainly does make free use of the mathematics of the infinite; irrational numbers, Minkowski space, partial differential equations, infinite dimensional Hilbert spaces, type III von Neumann algebras, even the axiom of choice is sometimes invoked. On the other hand, even without going as far as idealism, we should perhaps find the idea of a physical reality which is actually infinite somewhat disconcerting, in as far as we ourselves are finite beings, unable, for example, to make any measurements with unlimited accuracy. Moreover, because we are finite beings, any application we make of infinitary mathematics can always be replaced by an application of finitary mathematics with indiscernable consequences. This is important in mathematical physics, because it means that any significant consequence of infinitary mathematics must reflect facts that can be expressed with finitary mathematics. Often this is conceptually straightforward; as in the process of constructing a discrete version of a differential equation for numerical analysis. Sometimes, however, it is not immediately clear what sort of finitary fact might be involved and finding such a fact may be a valuable part of understanding the application of the mathematics.

As a functional analyst, I do not take kindly to being told, even by myself, that I should not talk about the Lebesgue integral or the axiom of choice. Thus, I shall not even begin to argue that infinitary mathematics is without validity. It is surely hard, for example, to doubt the truths of logical implication, in particular in as far as to doubt such implications is to doubt the means of expression which are required if doubt is to be meaningful. Our finite proofs give us adequate reason to believe that the axiom of choice is equivalent to Tychonoff's theorem and to Zorn's lemma. But, although, as we shall see, it may be useful to assume that arbitrary products of compact spaces are compact (Tychonoff's theorem), it does not necessarily follow that the axiom of choice is a necessary property of real infinite sets or, indeed, that there are any real infinite sets; any more than the fact that it is useful to know that the diagonal of a square of side a has length $a \times \sqrt{2}$ means anything more than that any real physical square has side a and diagonal d which satisfy $d^2 = 2a^2$ to a degree which depends on the extent to which the square approaches an unobtainable perfection. This line of argument makes it tempting to think that the mathematics of the infinite might be simply a language game; simply the study of proofs. The ultimate conclusion of this would be the idea that although finitary mathematics tells us necessary truths about physical reality, there might be no such necessary truths involving infinitary mathematics, because physical reality simply has no infinite aspects. This would not make, for example, theorems about the independence of the continuum hypothesis any less interesting as mathematics, but it would remove any remnant of the intuition that the continuum hypothesis must be true or false because of the actual existence of real physical continua.

It seems that the many good ideas in the philosophy of mathematics need to be re-worked again and again as we probe at the different levels at which mathematics is important. There are many excellent surveys of such ideas. I have enjoyed learning from and agreeing and disagreeing with those discussed in Chihara (1990), Hersh (1997), Burgess and Rosen (1997), and Maddy (1997). In particular, there is an interesting discussion of whether infinitary mathematics is indispensable in Maddy (1997, chapter II.6). In this paper, I shall probe mathematics at the level of a working mathematical physicist, and then at the level of physical law, and finally at the level of individual mental processes. The result will be to emphasize the gulf between finitary and infinitary mathematics and to relate that duality to dualities between events and their probabilities, and between experience and description, and even between realism and fantasy.

What is Mathematics About?

Whatever mathematics may be, it is constructed out of theorems. In this section and the next, therefore, I shall analyse some theorems in order to illustrate some ideas about the nature of mathematics and to demonstrate some of the quite subtle ways in which the mathematics of the infinite does arise in modern mathematical physics.

If asked to calculate 17 times 19 in your head, you might work it out from 17 times 20 minus 17, or, remembering that $(n-1)(n+1) = n^2 - 1$, you might realise that it was 18 squared minus 1, and then you might have forgotten the value of 18 squared, and would need to think of 4 times 81. You certainly wouldn't think about successors of 0, let alone about sets of sets of sets of the empty set.

Theorem One

$$17 \times 19 = 17 \times 20 - 17 = 18^2 - 1 = 4 \times 81 - 1 = 4 \times 80 + 4 - 1.$$

This theorem illustrates the fact that is no single or canonical path to mathematical truths. It may be possible to codify the logical structure of any theorem in set theoretic terms, but that is not the same as capturing the theorem's meaning. Theorem one can also be used to illustrate some of the ways in which mathematical facts can be represented and confirmed by physical processes. For example, the theorem could be confirmed by drawing, re-arranging, and counting dots on a page, or by using grains of rice on a table, or by using a calculator.

Theorem Two

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This beautiful result is frequently mentioned by philosophers of mathematics. It will be used here to provide further illustration of the multiplicity of meanings which can be attached to any mathematical truth. Indeed, there is multiplicity of meaning even in the symbol π used in the statement of the theorem. This symbol could be glossed as $I = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, which could in turn be considered as a line integral measuring distance along a set of points $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 = 1$. Considered as a Riemann integral, I has similar character to the convergent sum on the right hand side. π might alternatively be defined as the first strictly positive zero of the function $\sin x$ defined as $\sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$, or as $\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2$, or as $4(\sum_{n=0}^{N} \frac{(-1)^n}{2n+1} + (-1)^{N+1} \int_0^1 \frac{x^{2(N+1)}}{1+x^2} dx)$.

There is also a multiplicity in the techniques by which the theorem can be proved. A quantum mechanic would expand the function $\varphi(x) = x$ on $L^2[0, 1]$ in the orthonormal basis $(\sqrt{2} \sin n\pi x)_{n \geq 1}$, and obtain $\varphi = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n\pi} (-1)^{n+1} \varphi_n$, and hence $\frac{1}{3} = ||\varphi||^2 = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}$.

A pure mathematician might prefer a proof using uniform convergence rather than convergence in L^2. This can be done by using Fourier series techniques to prove that $\frac{1}{4}(x-\pi)^2 = \frac{1}{12}\pi^2 + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ on the closed interval $[0, 2\pi]$.

A more significantly different proof uses complex analysis and Liouville's theorem to show that $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in Z} \frac{1}{(z-n)^2} \sim \frac{1}{z^2} + \frac{\pi^2}{3} + O(z^2)$.

Another proof uses Cauchy's theorem and involves integrating the function $\frac{\log(1-z)}{z}$ around the half circle $\{z = re^{i\theta} : 0 \le r \le 1, 0 \le \theta \le \pi\}$ indented at $z = 1$. This proof can be modified and rewritten to use nothing beyond advanced calculus on real functions.

Just as the different methods for theorem one use different numbers, these proofs use different theories, different functional relations, different integrals, and different expansions. They all, however, ultimately require the idea of convergence as does the very statement of the theorem. The truth of theorem two can thus be seen as a fact of infinitary mathematics. However, it is also a succession of facts about approximations in finitary mathematics. For example, using the last of the proposed definitions of π, the theorem says that for any rational number $q > 0$, there is a natural number N_0 (which can be explicitly estimated) such that $N \ge N_0$ implies

$$|\frac{8}{3}(1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^N}{2N+1})^2 - (1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{N^2})| < q. \quad (1)$$

Facts of this sort can be confirmed by physical methods. Indeed, any inequality involving sums and differences of rational numbers can be converted into an inequality involving sums and differences of integers. It is possible to imagine this being checked directly using piles of pebbles. More practically, using a calculator to check the result will involve the performance of a long series of elementary electronic operations. Each of these operations is a physical model of a mathematical process on finite digit binary numbers.

Of course, theorem two might also be taken to express facts about physical circles; not only with π characterized as the ratio of circumference to diameter, but also, for example, with π characterized by the Buffon needle problem as equal to $2L/pD$ where p is the probability that a needle of length L hits a line when it is dropped in a suitably random way onto a floor marked with parallel lines separated by uniform distance D. General relativity tells us that, in general, such facts are actually false, but there are a wide variety of possible physical processes which could provide confirmations that with such characterizations of π the theorem is approximately true.

The multiplicity of ideas which can be expressed through theorems one and two suggests to me that, first and foremost, mathematics is about mathematics. In particular, I would apply this to the idea of mathematical existence. When we say, for example, that $\frac{\pi^2}{\sin^2 \pi z}$ has a second order pole at $z = 13$, we are using talk about the "existence" of a pole as a shortcut for talk about the consequences of that "existence"; for example, that $\frac{\pi^2}{\sin^2 \pi z} \sim \frac{1}{(z-13)^2}$ for $z \sim 13$ or that if we draw a diagram of the singularities, we had better, if our diagram goes that far, put a dot or a cross, or whatever symbol we want to use, at co-ordinates $(13, 0)$.

Idealism applies a similar idea of existence in scare quotes to physical objects. An idealist who says that a carelessly kicked stone can break a window, is using talk about "stones" as a shortcut for talk about the apparent consequences of the "existence" of "stones" – for example, the feeling of the "kick" and the sound of "breaking glass" and the subsequent fear or embarassment. An idealist turns talk about existence into talk about consequences and reduces all consequences to mental terms. In mathematics, it is sufficient to reduce all talk about consequences to talk about mathematics, and hence to talk about what mathematicians do or might do, and hence to talk about what they see themselves as "doing" or possibly "doing".

To justify idealism, it is necessary to explain the coherence of the world of appearance. In my many-minds interpretation of quantum theory, the explanation will involve the idea that minds are only likely to exist in as far as it is likely for rich patterns of information to develop, and that this is only possible in circumstances in which the stable repetition and development of patterns is likely.

The coherence of mathematics also needs to be explained. Explanations in terms of the realistic existence of mathematical objects may be rejected, on the grounds that it is neither clear how such existence could be constituted, nor how it could explain our knowledge. The fact that our understanding of the idea of π, for example, has so many different aspects should be seen not as a reflection of the existence of some perfect Platonic form, but rather of the importance of ideas about "circles" to beings who try to make sense of our sort of reality. In other words, our interest in rules which would govern ideal Euclidean circles, if they existed, does not imply that such circles do exist, but merely that it is possible and useful for us to make the abstractions which are needed to allow us to talk about them or to "posit" them (Quine 1951, §VI). In conventional, non-idealist, terms, this would be to say that imperfect circles do exist and that it is by trying to understand and simplify rules for imperfect circles that we come to discover rules for perfect circles. According to my interpretation of quantum theory, imperfect circles also do not exist. Instead, we exist as rich structured finite meaningful stochastic patterns which have high probability of short term continuation. It seems plausible that the simplest way in which that sort of existence is possible is if the patterns can give themselves some sort of geometric meaning. Ultimately, abstractions about ideal Euclidean circles can be made because collections of finitary facts like (1) are true. The possibility of and the motivation for discovering such truths are part of the structure of human reality at the deepest level.

Taming the Infinite in Mathematical Physics.

In this section, we shall consider three examples of the relation between mathematical physics and the infinite. Theorem three introduces a situation in which infinitary mathematics provides a model which is both convincing and clearly unrealistic. Theorem four seems, at first sight, to be both significant and necessarily infinitary. It therefore provides a challenging example of the question of whether any application of infinitary concepts can always be expressable in finitary terms. The final situation exemplifies the use of the axiom of choice in mathematical physics.

Theorem Three There are no phase transitions in finite systems.

One of the central problems of mathematical physics is to gain a theoretical understanding of the phases of matter – such as ice, water, and steam – starting from a quantum mechanical description of the electrons and nucleons involved. Much progress has been made by the analysis of model systems (Ruelle 1969, Bratelli and

Robinson 1981, Krieger, 1996). In such systems, a phase transition is defined as a singularity of a thermodynamic function, and there are models which do exhibit phase transitions of this nature which have properties which appear to reflect the properties of observed physical phase transitions. However, there are no such singularities in systems which model finite numbers of molecules. This means that, however realistic our models become, those phase transitions which satisfy the definition will only be a caricature of our observations. This is a strange situation in which our models appear to become more realistic, in that they describe changes of phase, only when they become less realistic, by describing infinitely many molecules.

There is no fundamental problem here, but although this is a case in which at least one infinitary aspect of the mathematics used is clearly not indispensable in that, of course, we cannot take the accuracy of this aspect of infinite models of physical systems as a demonstration that we are wrong to believe that Avogadro's number (a measure of the number of molecules in objects weighing, very roughly, a hundred grams) is actually finite, nevertheless it would be an over-simplification just to say that the mathematical definition of a phase transition is incorrect. In fact, we can never wait long enough for a real macroscopic system to reach a true equilibrium state; and anyway, for low temperatures, we could no more see such a state than we could see a cat being both alive and dead. On the other hand, the ergodic equilibrium states of infinite systems are actually excellent models of the observed quasi-equilibrium states of real (finite) macroscopic systems.

Theorem Four There are no pure normal states on a Type III von Neumann algebra.

This is a theorem which I believe may be of fundamental importance for the foundations of quantum theory in that it can be interpreted as saying that quantum theory should not ultimately be seen as a theory about wavefunctions. Wavefunctions are pure normal states on Type I von Neumann algebras, but there are good arguments for believing that local systems in quantum field theory should be described by Type III algebras instead (Haag 1992, §V.6). Quantum field theory is the most complete version of quantum theory available, and observers of course are localized systems. Type III von Neumann algebras, however, require infinite dimensional systems and this raises the question of how any useful aspects of this theorem might be expressed in finitary terms. Answering this question is not only a matter of soothing anxieties about the infinite but also of revealing the physics behind what might otherwise seem just mathematical name-dropping.

A caricature of the theorem is possible in commutative quantum theory; a subject which is sometimes referred to as "probability theory". On a finite set $\{1, \ldots, n\}$, a "state" is simply a probability distribution – a sequence $(p_i)_{i=1}^n$ satisfying $0 \leq p_i \leq 1$ for $i = 1, \ldots, n$, and $\sum_{i=1}^n p_i = 1$. A state $(p_i)_{i=1}^n$ is defined to be "pure" if and only if there is no decomposition into distinct states $(u_i)_{i=1}^n$ and $(v_i)_{i=1}^n$ such that, for $0 < x < 1$, $p_i = xu_i + (1-x)v_i$. It is quite easy to see that pure states exist and are exactly those probability distributions such that $p_i = 1$ for some i.

Now consider probabilities on an infinite set. On the real line $\mathbb{R}$, various types of probability distributions are possible, but the ones which correspond to normal states in quantum theory are the distributions defined by bounded measurable functions $p \in L^\infty(\mathbb{R}, dx)$ such that $0 \leq p(x)$ almost everywhere and such that the probability of $A \subset \mathbb{R}$ is given by $P(A) = \int_A p(x)dx$. This kind of state can never be pure, because it is always possible to find disjoint sets $A_1, A_2 \subset \mathbb{R}$ such that $P(A_1) = P(A_2) = \frac{1}{2}$ and then $p(x) = \frac{1}{2}(2I_{A_1}(x)p(x) + 2I_{A_2}(x)p(x))$ where I_A is the characteristic function of the set A. Those familar with the concepts in theorem four will prove it using a related idea – a pure normal state has a minimal support projection, but type III algebras have no minimal projections.

A pure state in quantum mechanics is a state which provides maximal information. In commutative quantum mechanics, pure states provide complete information (certainty). The non-existence of normal pure states in the infinite realm means that complete information is completely unobtainable. This is a good model of a situation in which it is difficult to obtain complete information. With classical mechanics on continuous space, it would have been absurd to try to find out by measurement whether the ratio of distances between two pairs of particles was a rational number. On the other hand, given a reasonable model of measurement, exact measurements on a discrete lattice may be expected to become increasing hard as the lattice spacing decreases. This means that from the point of view of a classical physicist who accepts that space is either continuous, or discrete at an imperceptibly short length scale, a theory, like quantum mechanics, in which the distance between a pair of particles cannot have an exact value is not obviously false.

In the application of theorem four, I would argue that we are in the situation that either wavefunctions do not exist for real localized systems, or that we cannot tell whether a quantum state on a real system is pure or mixed. In either case, a theory of quantum mechanics based on mixed states rather than wavefunctions cannot be obviously false.

Infinitary mathematics plays an important role in this sort of analysis. It introduces a conceptual issue in an extreme form ("pure states are impossible") and it also presents us with the mathematical challenge of working out how that issue can be expressed in approximations to the extreme situation. For example, if theorem four is physically significant, then it should tell us something physically significant about the nature of finite-dimensional quantum systems of large dimension. One possible statement of such a fact would be that the quantum entropy (S – a measure of the purity of a state) is hard to control in Hilbert spaces of large dimension (D). For example, the following proposition, which follows straightforwardly from the concavity of S and the existence of a state τ with $S(\tau) = \log D$, shows that in a Hilbert space of sufficiently large dimension, there is a state of large entropy close to any given state.

Proposition *Given $\varepsilon > 0$ and $M > 0$, there exists D such that, for any state ρ on a Hilbert space $\mathcal{H}$ of dimension at least D, there exists a state σ on $\mathcal{H}$ such that $||\rho - \sigma|| < \varepsilon$ and $S(\sigma) > \log M$.*

Note that this proposition is by no means entirely free of infinitary mathematics and neither is its interpretation entirely straightforward. The aim here, however, is merely to illustrate the fact that it would be a mistake in physics not to try to understand how any given invocation of infinitary mathematics could be expressed in finitary terms.

Theorem Five A product of compact spaces is compact.

This is Tychonoff's theorem, which, as I have mentioned, is equivalent to the axiom of choice. The various forms of the axiom of choice are useful in functional analysis because they can be used to prove that various "objects" "exist". For example, the Hahn-Banach theorem, which is proved using Zorn's lemma, shows the "existence" of extensions of linear functionals. Tychonoff's theorem can be used to demonstrate the "existence" of limits of generalized sequences. This is similar to the way in which the supposed completeness of the real numbers can be used to demonstrate the "existence" of a positive number r satisfying $r^2 = 2$.

The limits, however, can be considerably less innocuous than $\sqrt{2}$. For example, theorem four referred to "normal" states on a von Neumann algebra. On a finite-dimensional algebra, there are no other states, but in infinite dimensions, Tychonoff's theorem implies the w*-compactness of state space considered as a closed subset of the dual of the algebra. This means, for example, that if $(\psi_n)_{n=1}^\infty$ is an orthonormal basis for an infinite-dimensional Hilbert space $\mathcal{H}$, then the sequence $(|\psi_n\rangle\langle\psi_n|)_{n=1}^\infty$ of density matrices has a convergent subnet. In other words, there is a positive linear functional ρ on the space $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ such that, for all $A \in \mathcal{B}(\mathcal{H})$, $\rho(A)$ is a limit point of the sequence $(\langle\psi_n|A|\psi_n\rangle)_{n=1}^\infty$. In particular, $\rho(1) = 1$ and yet ρ is singular in the sense $\rho(|\psi_n\rangle\langle\psi_n|) = 0$ for all n. This shows that ρ is not σ-additive. In the language of probability theory, ρ defines a probability P on the positive integers which is additive but not σ-additive, and which vanishes on finite sets. However, P is far from completely defined by this statement. Given any infinite subset S of the positive integers, we can, for example, choose P to satisfy $P(S) = 1$. Then, given any infinite subset $S' \subset S$, we can choose $P(S') = 1$. Infinitely many distinct choices are required to specify P (or ρ). To provide a satisfactory mathematical theory of such states, and in particular to define pure singular states, it is necessary to use a form of the axiom of choice stronger than the axiom of dependent choice (Halvorson 2001).

It would be good to rule all non-normal states out of consideration in physical applications. This can be certainly done, as I have implied in discussing theorem four. Indeed, physical states can be assumed to have all sorts of physically-required properties in addition to normality, such as finite energy. Nevertheless, mathematical considerations can lead to the use of non-normal states in theorems of physical relevance. For example, in the development of a functional on pairs of quantum states which I believe to be relevant to the calculation of quantum probabilities, I invoked the w*-compactness of the space of states in order to give a simple definition in terms of suprema (Donald 1986, 1992). To make such a definition acceptable, it has to be accompanied by theorems showing that whenever non-normal states might arise, they have no necessary place in the physics (Donald 1986, Axiom IV, Donald 1992, Property K).

Infinite Approximately-Homogeneous Cosmologies.

There are many different aspects of physical reality which might be infinite. Some of these infinities could be more problematic than others. In particular, one idea which is often thought plausible in cosmology is that of a universe which is spatially or temporally infinite and which is approximately homogeneous. The sense of approximately homogeneous might quite reasonably be taken to be that there are infinitely many spacetime regions of the magnitude of our visible universe in which the same laws of physics apply and in which roughly the initial conditions of our big bang applied. In that case, we can apply the following theorems.

Theorem Six Let I be an infinite set and F be finite. Let $f : I \to F$. Then there are infinitely many elements of I which have the same image under f.

The theorem shows that if the universe contains infinitely many stars or planets or lifeforms or intelligent lifeforms or mathematical proofs, then there are infinitely many stars or planets or lifeforms or intelligent lifeforms or mathematical proofs which are arbitrarily similar by any finite measure. Given homogeneous laws of physics, a suitable finite measure might specify the approximate relative positions of a finite number of isotopically-identified species of individual atom over a finite sequence of times.

Theorem Seven Let $(A_n)_{n=1}^{\infty}$ be an infinite sequence of independent events for some probability P and suppose that, for some $\delta > 0$ and all n, $P(A_n) \geq \delta$. Then, with probability one, infinitely many of the A_n occur.

This theorem is a version of a Borel-Cantelli lemma. Applied to an approximately homogeneous infinite cosmology, it says that if it is possible for a particular star or lifeform or mathematical proof to exist in our visible universe, then, given the observed indeterminism of quantum processes, it is inevitable that infinitely many such stars or lifeforms or mathematical proofs exist which are arbitrarily similar by any finite measure. This is a sufficiently striking idea that it seems worth giving a proof of theorem seven.

Let X_N be the event that no A after A_N happens. X_N is less likely than that, for any M, A_{N+1} and A_{N+2} and ... and A_{N+M} don't happen, so that, using independence, $P(X_N) < (1 - \delta)^M$ for all M, and hence $P(X_N) = 0$.

Let X be the event that only finitely many of the A_n happen.

If X happens then, for some N, X_N happens, and so $P(X)$ is less than or equal to $P(X_1) + P(X_2) + P(X_3) + \dots$

Thus $P(X) = 0$, and theorem 7 is proved. ∎

The first part of this proof suggests that if you have to throw double six to get out of jail, then you will get out if you keep throwing long enough. Of course, this ignores the possibility that you might die first. The second part of the proof is the idea that the union of countably many events of probability 0 has probability 0.

It is sometimes argued that probability in a many-worlds interpretation of quantum theory is suspect in as far as such a theory is fundamentally deterministic with all possible future events actually happening in one world or another. Theorem seven shows that an approximately homogeneous infinite cosmology is somewhat similar. In neither situation is it entirely reasonable to say, when faced with a pair of distinct possible outcomes, "Either A or B, but not both is going to happen", because in both situations, nothing observed about the prior circumstances distinguishes between the apparently inevitable futures in which A and B occur. This apparent inevitability may however only be apparent in an idealist many-minds interpretation, such as my own, in which physical laws provide probabilities for minds to exist but it need not necessarily be the case that all possible consciousnesses are experienced.

Empirical Science and Infinitary Mathematics.

Even if our interactions with reality are entirely finite, and even if all of our observations can be explained in terms of finite models, it does not seem particularly plausible that they should be so explained. Euclidean geometry, Newtonian mechanics, thermodynamics, statistical mechanics, electromagnetism, special and general relativity, the Schrödinger equation, quantum electrodynamics, the standard model; the best empirically-validated theories, which have been used to explain and develop the technological achievements of our civilization, all depend on infinitary mathematics. Of course no theory comes with a guarantee of truth, and in fact, there are empirical gaps or failures in every known theory. In particular, we have yet to discover any coherent "theory of everything" which convincingly combines general relativity with quantum theory. It is just about conceivable that there might be a coherent and convincing theory of everything which depended entirely on finitary mathematics. But this seems pretty unlikely. Successful extensions and revisions of previous theories have always involved trying to work empirically derived concepts into simple and beautiful mathematical structures. Finitary mathematics however is rarely as simple, let alone as beautiful, as infinitary mathematics. It is easier, at least given appropriate training, to think of and to use $\sqrt{2}$ as a "number" rather than constantly to refer to any of the many ways of approximating that irrational by rationals. The continuous differential equations of mathematical physics are far more beautiful than the machine and code-dependent approximations to those equations which are used in the computation of numerical solutions. When cosmology turned away from infinite flat Euclidean geometry, it became necessary to wonder whether the universe might have a radius, and to ask how it might develop in time. If spacetime were actually a finite lattice of points, then questions about how many points there might be to a metre would arise. Is it appropriate to raise such questions without empirical input, just because of philosophical scruples about the mathematics of the infinite?

The crucial motivation for the application of infinitary mathematics in theoretical physics is the empirical success of theories developed to satisfy human taste in mathematical simplicity. What seems strange about this is that this simplicity is a simplicity of external description, which is not necessarily the same as an ontological simplicity. It is also not the same as a simplicity of total description.

Theorem Eight. Let I be an infinite set without distinguished elements. Then an infinite amount of information is required to identify any element of the set.

This theorem holds because if any point in I could be identified by a finite description, then the finitely many points which could be identified with the shortest description in a given language would thereby be distinguished. It might be relevant to the problem of specifying an individual star or lifeform or mathematical proof in an infinite approximately-homogeneous cosmology.

Mathematics, Madness, and Quantum Mechanics.

Quantum mechanics provides some important examples of the power of mathematical aesthetics to lead us towards ideas about the nature of reality, which, although compatible with empirical observation, are so far from common sense that they might be thought insane.

A lesser insanity, associated with the names of Einstein, Podolsky, Rosen, Bell, and Aspect, comes from taking too seriously a small part of the formalism of quantum theory and combining it with relativity theory (Bell 1987). This leads to a belief in spooky and mysterious connections between spatially distant points. For example, there are apparently situations in which one experimenter "Alice" can find out which of two genuinely random events a distant colleague "Bob" will observe, simultaneously with his observation, but without there being anything in Bob's laboratory which predetermines his result, or any message of any kind passed between the laboratories between Alice's observation and Bob's. This insanity unfortunately is essentially unavoidable as its consequences have been convincingly and directly demonstrated experimentally.

A greater insanity, from which I myself suffer in a particularly acute form, goes back to the idea of the Schrödinger cat experiment (Schrödinger 1935). What this thought experiment indicates is that, if the mathematical formalism of quantum theory, in the specific form of the Schrödinger equation, gives an accurate picture of the behaviour of ordinary macroscopic objects, then reality has to be pretty weird.

This weirdness was first taken at face value by Hugh Everett III (1957), who, in the development of the many-worlds interpretation, argued that the mathematics of quantum theory suggests that all possible future possibilities, even macroscopically distinct possibilities like whether a cat lives or dies, actually happen, but that being limited physical systems ourselves, we can only see individual possibilities.

Everett's work largely leaves open the questions of characterizing individual possibilities and our limited natures. My own approach to these questions has involved the analysis of minds as finite systems processing finite information. As minds, we appear to live inside physical reality, our brains apparently being direct physical representations of the structure of our minds. But, if minds are fundamental, then what we think of as "physical reality" is just a mental representation. What is "external" to mind, nevertheless, is the physical law which determines the probabilities of the possible futures of a given mental structure.

Although this picture is radical, it is, in my opinion, both logically consistent and consistent with empirical evidence. Moreover, it has considerable explanatory power. For example, mind is placed at the heart of reality, rather than being just an embarrasment as it would seem to be for materialists. Because time becomes an aspect of our individual structure as observers, the idea of a fixed observer-independent background spacetime becomes unnecessary. This may be useful in the development of quantum gravity theories. The problem of "fine-tuning" of physical constants can be resolved by noting that if all information is mental, then "constants" have to be determined by observation, are fixed only to the extent to which they have been observed, can only be observed if they have values which make observation possible, and are only likely to be observed if they have values which make observation likely. The situations in which one experimenter "Alice" can apparently find out which of two random events a distant colleague "Bob" will observe before his observation, but without there being anything in Bob's laboratory which predetermines his result, or any message of any kind passed between the laboratories between Alice's observation and Bob's, can be explained by noting that Alice cannot confirm that Bob's observations satify her predictions until the experimenters compare notes, and, at that stage, effectively-local physical laws can ensure that Alice's observations of Bob's results are compatible with the earlier observations from which she deduced her predictions. This explanation depends on Alice and Bob having equally fundamental but separate mental lives.

Dualities and Infinitary Mathematics in a Many-Minds Interpretation of Quantum Theory.

For the purposes of the present discussion, the most important aspect of my interpretation of quantum theory is the requirement that individual mental structures be finite. This does seem to be plausible. In particular, a fundamental discreteness does seem to be built into the actual operation of the human brain, at the level, for example, of local neural firing. It is also possible to make at least some sense of the stochastic nature of observed reality if that reality is like a finite game of chance. It seems to me far harder to understand what it would be like to be an observer if throws in the game of life allowed an infinite number of distinguishable possibilities each with infinitesimal probability. In such a situation, finite probabilities only exist for infinite sets of possibilities. Supposing that an observer cannot be aware of an infinite amount of information, it would seem not unreasonable to attempt to identify an observer with a finitely-definable equivalence class of observationally-indistinguishable possibilities.

As well as finite mental structures and discrete stochastic events, the theory also depends on physical laws. My theory is a form of idealism, in which the physical laws and initial conditions have no role or reality beyond that of providing probabilities for mental histories. These probabilities are defined through the construction of abstract sets of possible physical manifestations for each mental history. A mental structure is given a precise abstract definition and a "possible physical manifestation" is one of the ways in which the laws of physics can permit the development of something characterized as having that abstract structure. In the most complete current form of the theory (Donald 1999), these physical laws are taken to be the laws of special relativistic quantum field theory and the characterization allows for uncountably many possible manifestations for any given mental structure. In this way, the theory manages to push to a higher level of abstraction the problem of giving a precise characterization of an observable event; one of the most difficult problems in the interpretation of quantum theory. Infinitary mathematics is invoked by the bucketful in the technical appendix of Donald 1999, but that infinitary mathematics is invoked entirely in the process of defining probabilities for finite mental structures.

The gulf in a duality between experienced events and their probabilities is much wider than that in the conventional duality between mental experiences and the physical brains which are supposed to be possessed by those experiences. It is surely significant that infinitary and finitary mathematics lie on either side of this gulf, even if the eradication of the infinite remains incomplete. Physicists are familiar with a duality between initial conditions and dynamical equations. In classical physics at least, the dynamical equations are supposed to be simple and beautiful, while the initial conditions carry, once and for all, all the assumed infinite complexities of the assumed actual physical world. The duality I am proposing makes a considerable alteration to this picture. The initial conditions are replaced an entirely homogeneous initial state gradually and stochastically embellished by the finite developing complexities of an individual's actual observations up to that individual's current present. This, in fact, provides an explanation of the approximate homogeneity of our observed cosmos. The dynamical equations remain simple and beautiful, but infinitary mathematics is required in the analysis of the ways in which those complexities can arise and in the resulting actions of the equations.

Yet another related duality is that between "meaning" and "structure" or between "readers" and "texts". In a fascinating recent book, Tasić (2001) explains how versions of this duality have frequently appeared in the historical conversation between philosophy and mathematics. My work could be described, if I may be so bold, as "scientific" or "ontological" idealism rather than as "epistemological" idealism. This is to say that, in order to understand what physical theory and empirical observation seem to be telling us about the nature of reality, I have been led to propose a picture in which physical appearances are constructs of fundamental mental structures. My primary concern has been to use theory and observation to discover what form those structures might take and how they might develop, rather than to try to understand the process by which a given structure gives itself meaning or the reliability of the knowledge obtained in such a process. I am a realist about reality, but the form of the fundamental structures of reality is not immediately obvious to us, and our best hope of discovering that form lies in the scientific circle of experiment, interpretation, theory, aesthetics, mathematics, and consistency.

There are two ideas through which it might yet be possible to avoid the necessity of infinitary mathematics in the context of my interpretation, or an extension of my interpretation, of quantum theory. Neither idea, however, seems entirely satisfactory.

The first idea would avoid infinitary mathematics through the assumption that the ultimate correct theory of everything was an entirely finitary theory. It is and always will be possible to make such an assumption, but it is cheap for those not involved in the awesome difficulties of theories of fundamental interactions to presume to say how those theories should turn out. Moreover, the original starting point which eventually led me to this paper, lay not in a problem with infinitary mathematics as such, but rather in the gulf in complexity between the resources required to simulate a mental structure in my theory, and those required to simulate the probabilities for the development of that structure. With a finitary fundamental theory, that gulf will still be wide, even if it is no longer infinitely wide.

Given that in my proposals, infinitary mathematics is only required for the definition of finite sequences of probabilities, the second idea for avoiding the reality of infinitary mathematics would involve the fact that an individual finite sequence of events cannot precisely determine the probabilities of events making up that sequence. In this scenario, the gulf in resources required for a simulation would be narrowed by simulating probabilities only to a precision which would make individual structures likely to predict the correct theoretical probabilities. In Donald (2002), I argue that probabilities in a many-minds version of quantum theory need to be precisely defined in order to make sense of the idea of probabilities as theoretically-defined propensities. If the probabilities are not precisely defined, then the theory is incomplete. Nevertheless, events either happen or do not happen. The only empirical justification for accepting a probabilistic theory is that the events which have happened are typical events according to that theory. But a "typical event" is just an event which belongs to some class of significant sets which have high probability.

Theorem Nine Let $(A_n)_{n=1}^N$ be a finite sequence of events generating a σ-algebra $\mathcal{F}$ in a probability space $(X, \mathcal{F}, P)$. Then, given any $\delta > 0$, there exists an integer N and a probability Q, such that $Q(A) \in \{\frac{M}{N} : M = 0, 1, \ldots, N\}$ and $|Q(A) - P(A)| < \delta$ for all $A \in \mathcal{F}$.

example Suppose that a single sequence of N heads and tails is produced by a sequence of independent events with the probability of heads at each stage being $p = e^{-1}$.

Compare this with a single sequence of N heads and tails produced by a sequence of independent events with the probability of heads at stage n being $q_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

For the first process, the expected number of heads is $E^1(H) = e^{-1}N$ with a variance of $(e^{-1} - e^{-2})N$. For the second process, the expected number of heads is $E^2(H) = \sum_{n=1}^{N} \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

$$E^1(H) - E^2(H) = \sum_{n=1}^{N} \frac{(-1)^n(n-1)}{n!} + N \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!}.$$

$$|E^1(H) - E^2(H)| \leq \sum_{n=1}^{\infty} \frac{n}{n!} = e.$$

No possible statistical test can give a plausible argument to distinguish which method of generation has been used to produce a single such sequence, whatever the magnitude of N.

Ultimately, the generation of probabilistic events seems to be an entirely mysterious aspect of reality. Asking about it is like asking about where the initial conditions come from in a deterministic theory, or like asking why there is something rather than nothing. However the question of the extent to which infinitary mathematics is necessary in the generation of such events is somewhat more approachable. The intuition that I am working with here is a picture of reality as being furnished with a random number generator which provides, or has provided, a sequence of random events of fixed probability – say a sequence of bits of probability $\frac{1}{2}$. Theorem nine indicates that each of the random events which make up an individual's experiences could then be determined by a finite number of those bits which could provide a choice indistinguishable to that individual from the ideal probability generated by the infinitary mathematics. In this way, even the probabilistic generation of an individual's experiences might only involve a finite complexity directly reflecting the complexity of the experiences.

Nevertheless, although it might theoretically be possible in this way to avoid any actual infinitary mathematics in the construction of our reality, it would be completely unscientific to ignore the manifest simplicity of the physical laws which seem to govern that reality. Indeed, surely simplicity must underlie any explanation for the monotonous regularities of the cosmos. Galaxies, and lives, and suffering seem cheap in the economics of creation; outcomes and purpose seem expensive. We appear to ride the crest of an evolutionary wave which formed us merely because our ancestors happened to be able to survive. My interpretation of quantum theory amounts to the definition of a stochastic process on abstract patterns of information. Despite the complexity of this definition, it seems to me far from inconceivable that among all the possible ways in which stochastic processes can be defined on such patterns, the postulated definition is actually as simple as any comparable set of rules which make likely rich and meaningful patterns of information. "Simplicity" in this context, however, is the simplicity of infinitary mathematics, which is a simplicity of external description rather than a simplicity of construction. Is my desire to give an explanation compatible with the latter as well as the former merely yet another anthropocentric mistake about the economics of creation?

These remarks seem excessively speculative. Perhaps any attempt at a thorough analysis of the nature of reality is bound to end in speculation and mystery, but at least speculation is better than dogma. In the next section, we shall leave the mysteries of the generation and meaning of probabilities and return to our world of finite experiences to consider similarities between the fundamental structures of mind, language, and finitary mathematics.

The Possibility of Possibilities.

Mind, language, and finitary mathematics are mutually fundamental and each depends on the possibility of possibilities. Language represents meaning through finite structured patterns. Mathematics is the study of structured patterns. Mind is a finite structured pattern which discovers its own meaning through being the history of its development. Structures carry information only because they could be other than they are. Words are only interesting if something else might have been said. Mind is awareness of external reality as external. It is awareness of possibilities of pain and of comfort, of hunger and of satisfaction, of interest and of boredom. My characterization of an "observer" as a finite information-processing structure requires an analysis of such a structure as a pattern of repetitions and denials of abstractions of elementary localized events. For such a pattern to be meaningful, it must recognise itself as a carrier of information in the same way that the meaning of a language is recognized by its speakers.

A reality in which meaningful repetition is possible is a reality in which finitary mathematics is possible. Counting is built on the abstraction and repetition of a meaningful property, like "being an apple". An "apple" is a "not-orange", but it is also any apple. Meaning is built on the representation of repeatable aspects of (apparent) reality which do not capture the totality of that reality. An "apple" is a weight like this, and a shape like this, and a smell like this, and a colour like this (or this, or this), and a taste like this. "Like" refers to a broad and imprecise range because, in biological terms, it would be both impossible and counter-productive to sense weights or shapes or smells or colours or tastes exactly. We are able to count apples, because it is possible, usually, for us to recognise an apple when we see one, and to expect it to behave like a solid object as we move it around. An apple doesn't disappear without the possibility of an explanation, such as that it has been eaten or stolen or hidden. Finitary mathematics tells us how collections of objects that tend not to disappear must behave when the objects don't disappear. For example, $17 \times 19 = 18^2 - 1$ because any collection of 324 such objects can be moved from a square array of side 18 to a rectangular array of sides 17 and 19 with 1 left over. Marks on paper also tend not to disappear, so that we can draw pictures and, eventually, develop a symbolism which allows us to represent arithmetical laws which would apply to any finite collection of stable objects.

The point I want to make here is not the dogmatic claim that finitary mathematics is just a collection of facts about possible collections of "objects", but rather that stability and possibility and abstraction and difference are utterly fundamental not only to the meaning of mathematics, but also to the meaning of language, to the structure of our minds, and to our ability to make sense of reality. Mathematics is no more just a language game than language is just a mathematics game. I say tomahto and you say tomayto, but what is essential is that neither of us say grapefruit. Language depends on abstractions and representations and rules. Language is possible if and only if finitary mathematics is possible. Mind too, characterized as a pattern of information defined by rules representing elementary events which occur and change and recur, I believe to be possible if and only if finitary mathematics is possible.

Possibility is fundamental to our reality, so it is not surprising that there are many challenging metaphysical issues associated with it. Among those relevant to the present discusion are whether to be possible is to be; how far possibility extends beyond "reality"; and the relations between possibility, truth, and fantasy, and between the finitary and the infinitary.

In a philosophical analysis of the nature of possibility, Lewis (1986) attempts a defense of the doctrine of modal realism – the idea that to be possible is to be. Modal realism and mathematical realism have a good deal in common. A central question in both cases is how our thoughts seem able to go beyond what we are, and the proposed answer in both cases is that our thoughts can go beyond what we are because what there is goes beyond what we are. Both cases meet similar problems with the infinite, with multiplicities of variant descriptions, and with the question of how we can have knowledge of realities from which we are isolated. Modal realism has the additional problem that without some sort of a priori measure on the space of possibilities, it seems hard to justify empirical induction and to explain the apparent simplicity of our observed reality. On the other hand, I have a physical theory which allows me to define precisely what I believe an observable possibility to be and how likely each possibility is, but I do not believe that whether or not every one of these possible experiences corresponds to an experienced reality makes any difference to the experiences of those realities which are experienced. This is just to say that, as with the problem of solipsism, no individual mind in a "many-minds" theory can ever be certain, either from observing or from infering the behaviour of other bodies, that other minds are real. Indeed, because of this, leaving aside historical reasons and reasons to do with the nature of quantum dynamics and how that dynamics is used to calculate probabilities, it would make as much sense for me to refer to my "many-minds" interpretation of quantum theory as a "possible-minds" interpretation.

Related to the idea that possibilities cannot be understood unless they are in some sense real, is the idea that the meaning of a number depends on the actual physical existence of that many distinct entities, and so that there might be a largest actual finite number corresponding, in some way, to the number of entities in the universe. Even if I believed the physics behind this claim, and of course I do not, it would not seem to me that altering the actual size of the physical universe could make any difference to the truth of any mathematical statement. Nor do I accept Rotman's (1993) arguments that we should limit our imaginations by our apparent physical abilities.

The meanings of $(2816)^{71} + (2793)^{71} \neq (2834)^{71}$ include not only facts about parity, but also facts about what would happen if hyperbeings in $71 + 1$-dimensional spacetime tried re-arrangements of hyperobjects arranged in hypercubes. Given that we are not in contact with them, whether there really are such beings is irrelevant to our statements about what would happen if there were and to the meaning of the formula for us. Our minds always go beyond what we are (Sartre 1943). But they do not go beyond what we can imagine; in other words, they do not go beyond the partial representations that we can construct starting from our representations of what we are and what we are not. We are frustrated when our train is late, not because there is a world somewhere in which the train is on time, but because we have seen the timetable and know what we are entitled to expect. Mathematics is an extension of the framework of the constructions of our imagination. Mathematics builds into itself and makes explicit the laws of possibility – the laws of logically consistent construction which allow us to distinguish between possibility and impossibility.

All our lives we try to tame reality by telling stories. Language, mind, and mathematics are all part of this process. For mind to be possible there have to be stories to be told. Mathematics is first and foremost about mathematics, in the same way that "The Lord of the Rings" is first and foremost about hobbits – we can tell stories which go way beyond representations of our immediate biological needs and which build on their own internal structures.

Fantasies are unrealistic stories. To ride off into the setting sun is possible; to live happily ever after is a fantasy. The gulf between infinitary and finitary mathematics is sufficiently wide that there is a sense of "realistic" according to which the processes of infinitary mathematics are fantasies, but any finitary process in finitary mathematics is not. To match a rectangle of $(10^{436}-1)\times(10^{436}+1)$ physical objects with a square of side 10^{436} less one would be possible if it were possible for so many objects to be handled, but to match *every* positive integer with its double is a fantasy. Finitary mathematics is concerned with possibilities which are straightforward extensions of the possibilities within which we live our mental lives. There is no natural boundary within the finite numbers at which any "largest number" could plausibly be located. ∞, however, does make a fundamental boundary. I accept rules that would govern arrangements of hyperobjects by hyperbeings in $71+1$-dimensional spacetime for the same reasons that I accept rules governing arrangements of grains of rice on a table. These rules depend on logical consequence together with the idea of the predictable stability of physical "objects". The most important aspect of these finite possibilities is that what would be involved in their achievement can be completely understood. After all, it seems about as likely that I will spend much time arranging rice as that I will be contacted by hyperbeings, and I don't have a horse.

As stories, fantasies can certainly be simple, coherent, and meaningful. We surely know what it means to match every positive integer with its double. "Dot, dot, dot" is an essential part of the abstraction which is fundamental to our mental processes. We all intend to live happily ever after. We can only realize our dreams one day at a time, but we rarely know when the end will come. If n days are possible, then $n+1$ days are possible, so, ignoring probability, we dream of forever. Infinitary mathematics is a fantasy world in which we fantasize about the completions of processes which, realistically, we can only begin. When we write

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}+\frac{\pi^2}{24}$$

and deduce that $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8},$$ we need some experience in the manipulation of infinite series to be sure that what we are doing is justified. That experience can be expressed by the ε and δ arguments which reduce the manipulations to infinite classes of possible manipulations involving finite sums. We also need to justify the claim that $\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8}$. That justification ultimately comes from experience with integers, with rationals, with finite sequences, and with the logic of finite sentences and classes of sentences such as (1). Even if the laws of nature are statements of infinitary mathematics, our understanding of those laws remains founded on our finite experiences and it remains a fantasy to imagine the processes of infinitary mathematics being realized by finite beings.

Conclusion.

Mathematics is about mathematics, but this is not to say that it is just a social practice, or just a language game. It is both, but it is also a means of discovering truth. It is useful to talk as if mathematical objects exist, just as it is useful to talk as if characters in stories, or colours, or physical objects exist. The gulf between infinitary and finitary mathematics is fundamental. We are finite beings and the rules which structure our minds, our languages, and our lives are the same as the rules which underlie the truths of finitary mathematics. We can always avoid the use of infinitary mathematics in our scientific theories, but the cost may be the simplicity of description which has led us to those theories.

As finite beings, the simplicity of infinitary mathematics is, for us, the simplicity of fantasy. As mathematicians, we build these fantasies; as physicists, we try to grasp the fantasies which may actually govern our reality; and as philosophers, we can fantasize about understanding it all.

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